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Fractality and the small-world effect in Sierpinski graphs

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Abstract

Although some real networks exhibit self-similarity, there is no standard definition of fractality in graphs. On the other hand, the small-world phenomenon is one of the most important common properties of real interconnection networks. In this paper we relate these two properties. In order to do so, we focus on the family of Sierpinski networks. For the Sierpinski gasket, the Sierpinski carpet and the Sierpinski tetra, we give the basic properties and we calculate the box-counting dimension as a measure of their fractality. We also define a deterministic family of graphs, which we call small-world Sierpinski graphs. We show that our construction preserves the structure of Sierpinski graphs, including its box-counting dimension, while the small-world phenomenon arises. Thus, in this family of graphs, fractality and small-world effect are simultaneously present.

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1. Introduction

In the past ten years, complex networks have received the attention of researchers from a variety of areas. The ability to capture, store and process huge amounts of information allows the quantitative analysis of large, apparently random and unpredictable systems. Parallel research developments have shown that biological, social and economic networks, communication networks, the Internet and the www, as well as other artificial systems such as software architecture networks, are far from random and share interesting properties [1–3].

Three properties—small-world effect, scale-free degree distribution and hierarchical modularity—constitute the basis of our understanding of network organization. A small-world network is a network with a small average distance (or small diameter) and a large clustering. The scale-free degree distribution refers to the high degree of heterogeneity in the connectivity patterns. Moreover, some networks exhibit a large-scale organization characterized by a hierarchical modularity. Not surprisingly, some papers focus on how these properties are related. For instance, the relation between large-scale organization (i.e., hierarchical modularity and scale-free) and local subgraph density has been studied in [4]. In [5], the authors show that scale-free and small-world are not independent. A new perspective was raised by Song, Havlin and Makse in [6], where the self-similarity of important topologies is highlighted. Roughly speaking, the authors show that in some complex non-homogeneous networks, a measure of fractality can be calculated by adapting the classical box-counting method. The introduction of this new tool in the analysis of real networks allowed these authors to show the scale invariance of many real networks such as the www, cellular networks, protein-interaction networks, etc. However, the boxes considered in their method are not directly associated with the possible modularity of the network, and self-similarity appears when the network is 'renormalized' with the boxes converted into new nodes of a smaller network during a coarsening process. Their study suggests that real networks have evolved through an optimization process to a critical state with fractality being its telltale sign. They also note that some standard models of scale-free networks, such as the Barabási–Albert preferential attachment model [7], are not self-similar when considering this new measure. In a recent paper [8], the same authors show that the emergence of a self-similar fractal network is favoured by a diassortative process in its growing evolution for which high degree nodes prefer connections to low degree nodes. Moreover, they also show that a fractal network topology is inherent to the evolution of robust networks based on functional modules.

It is of interest to study these important properties of real networks with deterministic exact methods. Deterministic models have the strong advantage that it is often possible to compute analytically their characteristics, which may be compared with observational and experimental data from real and simulated networks. With this motivation, we introduce here a deterministic family of graphs, small-world Sierpinski graphs, which show simultaneously fractality and the small-world effect.

A lot of research has been done on small-world networks since they were characterized in [9], either unveiling the small-world structure of specific real networks or proposing mathematical models for small-world networks. A survey on such networks and how to model them can be found in [10]. Usual models are based on the existence of an underlying structure with some extra links, added via a certain augmentation process. In [11] a construction of deterministic small-world networks is given. From an algorithmic point of view, a small-world graph can be seen as a graph in which very short paths between nodes can efficiently be found, with no global knowledge [12–14].

On the other hand, the concepts of fractality and self-similarity are closely related to the concept of dimension, which is not easily definable for graphs. A graph $G = (V, E)$ consists of a non-empty set V of elements called vertices or nodes and a set E of pairs of elements of V called edges. A mathematical definition of self-similar graph can be found in [15]. As far as the authors know, there is no standard definition of fractality for graphs. Moreover, apart from the research on fractality in network traffic and the recent papers from Song, Havlin and Makse very few articles deal with fractality in networks. The work [16] by Goh *et al* is worth mentioning, where the analysis of topological fractality in a graph is related to the study of a specific spanning tree, the so-called skeleton.

In order to study a possible relation between the small-world effect and fractality, while seeking a better understanding of fractal network topology, we focus on the family of Sierpinski graphs. These graphs, derived from the Sierpinski triangle [17], have been considered from a probabilistic point of view. In particular, the problem of random walks in Sierpinski graphs has extensively been studied (see references in [15]). Here we adopt a combinatorial approach.

Our results. We present the basic properties of three families of Sierpinski graphs: Sierpinski gasket, SG_n, Sierpinski carpet, SC_n, and Sierpinski tetra, ST_n, where $n \in \mathbb{N}^+$. We also deterministically construct three families of small-world Sierpinski graphs: $SWSG_{n,m}$, $SWSC_{n,m}$, and $SWST_{n,m}$, for $n \geqslant 3$ and $2 \leqslant m \leqslant n-1$.

In both cases, Sierpinski graphs and small-world Sierpinski graphs, we compute the diameter, the clustering[1] and the box-counting dimension. For the computation of the box-counting dimension we apply the box-counting method, which is explained in detail in [6]. Indeed, in all cases, the small-world Sierpinski graphs have the same box-counting dimension as their corresponding 'not small-world' Sierpinski graphs.

Moreover, we show that small-world Sierpinski graphs are small-world, in the sense that their diameter, D, is logarithmic in the order, N. More precisely, we show that

- if $m \leqslant \log_2 n$, then $SWSG_{n,m}$ satisfies $D = O(\log N)$;
- if $m \leqslant \log_3 n$, then $SWSC_{n,m}$ satisfies $D = O(\log N)$;
- if $m \leqslant \log_2 n$, then $SWST_{n,m}$ satisfies $D = O(\log N)$.

Finally, we show that the construction of small-world Sierpinski graphs from Sierpinski graphs does not imply a great change in the graph structure, that is, the number of added links is small with regard to the number of links of the graphs and so is the clustering variation. Specifically, a comparison between Sierpinski and small-world Sierpinski graphs gives that the number of added edges is $O\left(\frac{N}{\log N}\right)$ and the clustering variation is $O\left(\frac{1}{\log N}\right)$ in the following cases:

- $SWSG_{n,m}$, with $m = \log_3 n$;
- $SWSC_{n,m}$, with $m = \log_3 n$;
- $SWST_{n,m}$, with $m = \log_4 n$.

All of these graphs have diameter $O(\log N)$.

For the sake of completeness we include the family of Sierpinski carpet and small-world Sierpinski carpet. This allows us to give a complete view of the Sierpinski family. However, their clustering coefficient is 0, so they cannot strictly be called small-world in the usual sense.

2. Sierpinski graphs

In this section we present the basic properties of Sierpinski graphs. This family of graphs comes from the Sierpinski gasket, the well-known fractal object introduced by Sierpinski in 1915 [17]. This graph family includes the Sierpinski gasket (see figure 2), the Sierpinski carpet (see figure 4) and the Sierpinski tetra (see figure 6). All these graphs can be seen as the underlying graph of its corresponding original fractal object. Our work focuses on their properties as models for networks.

Sierpinski graphs are recursively constructed from a basic building block. In the case of Sierpinski gasket, we start with a triangle, SG_1. Let us denote by v_T, v_L and v_R the nodes of SG_1. SG_2 is derived from three copies of SG_1, denoted by B^T, B^L and B^R, and identifying the three pairs of nodes: $v_T^L \equiv v_L^T$, $v_T^R \equiv v_R^T$, and $v_R^L \equiv v_L^R$ (T, L and R stands for 'top', 'left' and 'right'). In the natural planar representation of SG_2, we can also distinguish three nodes, the vertices of the external triangle, that can also be denoted by v_T, v_L and v_R.

For every $n \geqslant 2$, the graph SG_n is constructed from three copies of SG_{n-1} by identifying three pairs of distinguished nodes, as we did for SG_2. Again, SG_n has a natural planar representation and three distinguished nodes in the external face, which is represented as a

[1] The node clustering is defined as $\frac{2e}{d(d-1)}$, where d is the degree of the node and e is the size of the subgraph induced by its neighbours. The clustering of a graph is the average of the node clustering over all the nodes.

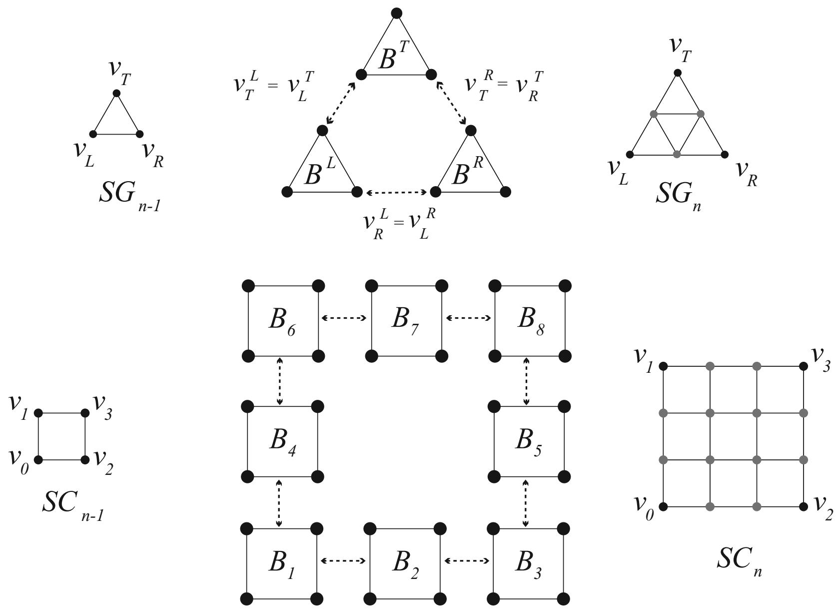

Figure 1. The recursive constructions of SG_n and SC_n, with all the node identifications.

triangle. Similar recursive constructions give the Sierpienski carpet and the Sierpinski tetra. In the case of Sierpinski carpet, the basic building block is a square. SC_n is derived from eight copies of SC_{n-1}, which are assembled according to a 3×3 grid, with no central element. Neighbouring blocks share a path. SC_n has a natural planar representation in which the external face is represented as a square, and the four vertices of this external square are the four distinguished nodes of the graph. For the Sierpinski tetra, the basic building block is a tetrahedron and, thus, we use a three-dimensional representation. ST_n is derived from four copies of ST_{n-1}, in this case assembled according to the vertices of a bigger tetrahedron, in a way analogous to the construction of SG_n. The four distinguished nodes are the nodes in the graph placed at the vertices of the tetrahedron in $\mathbb{R}^3$ which the graph is inscribed in. Figure 1 shows a graphical representation of this construction for SG_n and SC_n. This construction will allow us to prove the results in the following three sections.

2.1. Sierpinski gasket

Proposition 1. *The Sierpinski gasket, SG_n, satisfies the following properties (see figure 2):*

(i) the order and the size of SG_n are $|V_n| = \frac{3^n+3}{2}$ and $|E_n| = 3^n$;
(ii) the diameter of SG_n is $D_n = 2^{n-1}$;
(iii) the clustering of SG_n is $C_n = \frac{4 \cdot 3^{n-2}+5}{3^n+3}$.

Proof. Let a_n and b_n be $|V_n|$ and $|E_n|$, respectively. By construction of SG_n, for every $n \geqslant 2$, $a_n = 3a_{n-1} - 3$ and $b_n = 3b_{n-1}$. Moreover, $a_1 = 3$, and $b_1 = 3$. By solving the recurrence equations we obtain $a_n = \frac{3^n+3}{2}$ and $b_n = 3^n$.

It can easily be seen that $D_n = 2D_{n-1}$. Since $D_1 = 1$, we have $D_n = 2^{n-1}$.

The vertices of the external triangle are the only nodes of degree 2. Those three nodes have clustering 1. For $n \geqslant 2$, all the remaining nodes have degree 4, and the clustering is either $\frac{1}{2}$ or $\frac{1}{3}$. Let x_n be the number of nodes in SG_n with clustering $\frac{1}{2}$, and y_n be the number

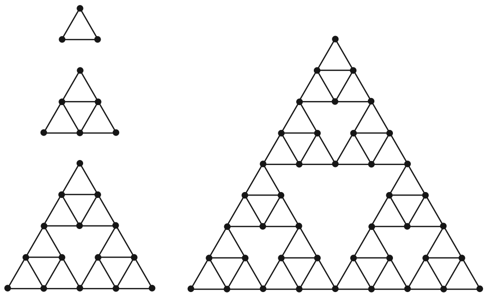

Figure 2. SG_1, SG_2, SG_3 and SG_4.

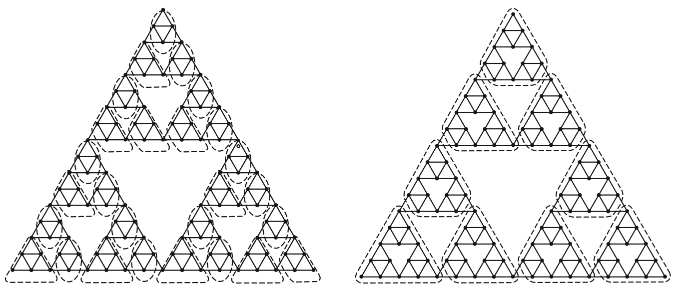

Figure 3. Boxes in SG_5 with $\ell_B = 3$ (distance = 2) and $\ell_B = 5$ (distance = 4).

of nodes in SG_n with clustering $\frac{1}{3}$. The recursive construction of SG_n allows us to derive the following recurrence equations:

$$x_n = 3x_{n-1}, \qquad x_2 = 3$$
$$y_n = 3y_{n-1} + 3, \qquad y_2 = 0.$$

The solution of this system is $x_n = 3^{n-1}$ and $y_n = \frac{3^{n-1}-3}{2}$. This gives us the value for the clustering

$$C_n = \frac{3 + \frac{1}{3} \cdot \frac{3^{n-1}-3}{2} + \frac{1}{2} \cdot 3^{n-1}}{\frac{3^n+3}{2}} = \frac{4 \cdot 3^{n-2} + 5}{3^n + 3}.$$

□

Proposition 2. *The box-counting dimension of SG_n is $d_B = \frac{\ln 3}{\ln 2} \approx 1.585$.*

Proof. According to the box-counting method, the box-counting dimension d_B is given by $N_B \approx \ell_B{}^{-d_B}$, where N_B is the number of boxes of linear size ℓ_B needed to cover the graph. The linear size of a box is one plus its diameter.

For different values of ℓ_B, we cover SG_n with boxes of linear size ℓ_B and we count the number of boxes (see figure 3). The found values are shown in the following table.

l_B	2	3	5	$\cdots$	$2^{n-1}+1$
N_B	3^{n-1}	3^{n-2}	3^{n-3}	$\cdots$	1

The value obtained for the box-counting dimension is $d_B = \frac{\ln 3}{\ln 2} \approx 1.585$. □

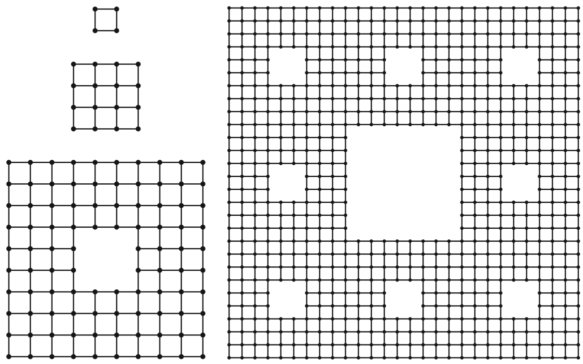

Figure 4. SC_1, SC_2, SC_3 and SC_4.

2.2. Sierpinski carpet

Proposition 3. *The Sierpinski carpet, SC_n, satisfies the following properties (see figure 4):*

(i) the order and the size of SC_n are $|V_n| = \frac{11}{70} \cdot 8^n + \frac{8}{5} \cdot 3^{n-1} + \frac{8}{7}$ and $|E_n| = \frac{3}{10} \cdot 8^n + \frac{8}{5} \cdot 3^{n-1}$;
(ii) the diameter of SC_n is $D_n = 2 \cdot 3^{n-1}$;
(iii) all the nodes in SC_n have clustering 0.

Proof. Let a_n and b_n be $|V_n|$ and $|E_n|$ respectively, and let ℓ_n be the side length of the external square in SC_n. By construction of SC_n, the following recurrence equations hold:

$$\begin{aligned} \ell_n &= 3\ell_{n-1} - 2, & \ell_1 &= 2 \\ a_n &= 8(a_{n-1} - \ell_{n-1} - 1), & a_1 &= 4 \\ b_n &= 8(b_{n-1} - \ell_{n-1}), & b_1 &= 4. \end{aligned}$$

By solving this system we obtain $\ell_n = 3^{n-1}$, $a_n = \frac{11}{70} \cdot 8^n + \frac{8}{5} \cdot 3^{n-1} + \frac{8}{7}$, and $b_n = \frac{3}{10} \cdot 8^n + \frac{8}{5} \cdot 3^{n-1}$.

It can easily be seen that D_n doubles the length of the side of the external square, i.e., $D_n = 2\ell_n = 2 \cdot 3^{n-1}$.

Since SC_n has no triangles, the clustering is 0. □

Proposition 4. *The box-counting dimension of SC_n is $d_B = \frac{\ln 8}{\ln 3} \approx 1.8928$.*

Proof. The proof is analogous to the proof of proposition 2. For different values of ℓ_B, we cover SC_n with boxes of linear size ℓ_B and we count the number of boxes (see figure 5). The values obtained are shown in the following table.

l_B	3	7	19	$\cdots$	$2 \cdot 3^{n-1} + 1$
N_B	8^{n-1}	8^{n-2}	8^{n-3}	$\cdots$	1

The value obtained for the box-counting dimension is $d_B \approx 1.8928$. □

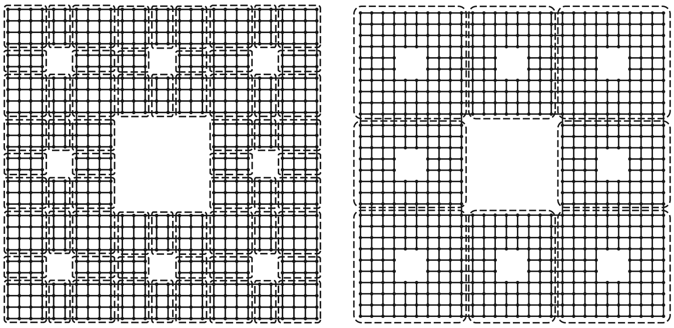

Figure 5. Boxes in SC_4 with $\ell_B = 7$ (distance = 6) and $\ell_B = 19$ (distance = 18).

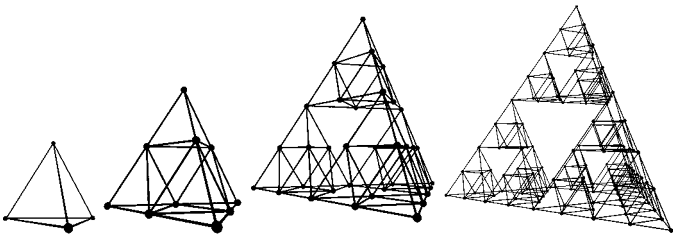

Figure 6. ST_1, ST_2, ST_3 and ST_4.

2.3. Sierpinski tetra

Proposition 5. *The Sierpinski tetra, ST_n, satisfies the following properties (see figure 6):*

(i) the order and the size of ST_n are $|V_n| = 2(4^{n-1}+1)$ and $|E_n| = 6\cdot 4^{n-1}$.
(ii) the diameter of ST_n is $D_n = 2^{n-1}$.
(iii) the clustering of ST_n is $C_n = \frac{5\cdot 4^{n-1}+16}{10(4^{n-1}+1)}$.

Proof. Let a_n and b_n be $|V_n|$ and $|E_n|$, respectively. By construction of ST_n, for every $n \geqslant 2$, $a_n = 4a_{n-1} - 6$ and $b_n = 4b_{n-1}$. Moreover, $a_1 = 4$, and $b_1 = 6$. By solving the recurrence equations we obtain $a_n = 2(4^{n-1}+1)$ and $b_n = 6\cdot 4^{n-1}$.

As we have shown for the Sierpinski gasket, $D_n = 2D_{n-1}$. Since $D_1 = 1$, we have $D_n = 2^{n-1}$.

The only nodes of degree 3 are the four vertices of the external tetrahedron. These four nodes have clustering 1. For $n \geqslant 2$, the clustering is either $\frac{8}{15}$ or $\frac{6}{15}$. Let x_n be the number of nodes in ST_n with clustering $\frac{8}{15}$, and y_n be the number of nodes in ST_n with clustering $\frac{6}{15}$. The recursive construction of ST_n allows us to derive the following recurrence relations:

$$x_n = 4x_{n-1}, \qquad x_2 = 6$$
$$y_n = 4y_{n-1} + 6, \qquad y_2 = 0.$$

The solution of this system is $x_n = 6\cdot 4^{n-2}$ and $y_n = 2(4^{n-2} - 1)$. This implies that the clustering is

$$C_n = \frac{4 + \frac{6}{15}\cdot 2(4^{n-2}-1) + \frac{8}{15}\cdot 6\cdot 4^{n-2}}{2(4^{n-1}+1)} = \frac{5\cdot 4^{n-1}+16}{10(4^{n-1}+1)}.$$

□

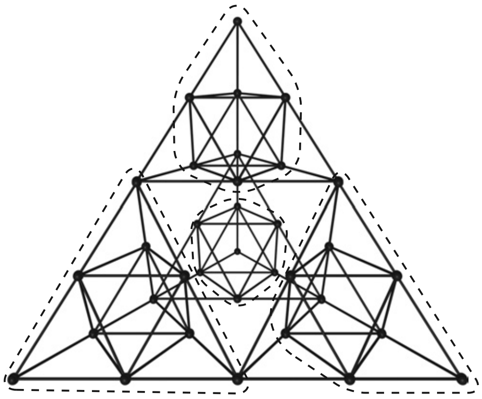

Figure 7. Boxes in ST_3 with $\ell_B = 3$ (distance= 2).

Proposition 6. *The box-counting dimension of* ST_n *is* $d_B = 2$.

Proof. The proof is analogous to the proofs of propositions 2 and 4. For different values of ℓ_B, we cover ST_n with boxes of linear size ℓ_B and we count the number of boxes (see figure 7). The values obtained are shown in the following table.

l_B	2	3	5	$\cdots$	$2^{n-1}+1$
N_B	4^{n-1}	4^{n-2}	4^{n-3}	$\cdots$	1

The value obtained for the box-counting dimension is $d_B \approx 2$. □

3. Small-world Sierpinski graphs

This section is devoted to the construction of small-world Sierpinski graphs.

Our goal is to reduce the diameter enough so as to attain a logarithmic diameter, while maintaining the original graph structure. For the three previous families of Sierpinski graphs, we propose similar constructions based on the adding of a node linked to a certain set of original nodes.

As we will show in section 4, both the number of added edges and the clustering variation are reasonably small.

3.1. Small-world Sierpinski gasket

The small-world Sierpinski gasket, $SWSG_{n,m}$, is the graph defined as follows.

For every $n \geqslant 3$ and $m = 2, \ldots, n-1$, SG_n can be seen as 3^{n-m} copies of SG_m, with some node identifications. $SWSG_{n,m}$ is the graph obtained by joining a new node to every vertex of the external triangle of every copy of SG_m (see figure 8). The number of new edges is exactly the number of vertices of SG_{n-m+1}.

The diameter of this new graph depends on the value of m. We will show that, for some of the values of m, $D(SWSG_{n,m}) = O(\log |V(SWSG_{n,m})|)$. Thus, this construction gives us a graph which is a small-world fractal.

Next we give the properties of $SWSG$.

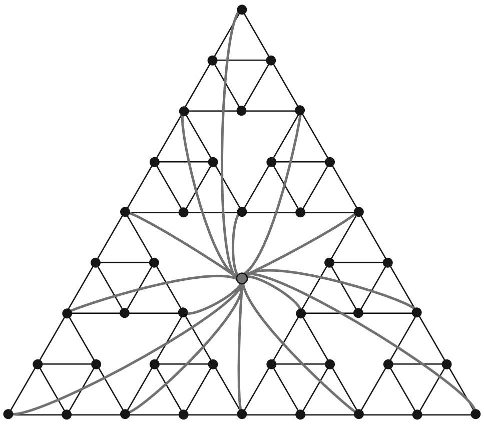

Figure 8. $SWSG_{4,2}$.

Proposition 7. *The small-world Sierpinski gasket, $SWSG_{n,m}$, satisfies the following properties (see figure 8):*

(i) the order and the size of $SWSG_{n,m}$ are $|V_{n,m}| = \frac{3^n+5}{2}$ and $|E_{n,m}| = 3^n + \frac{3^{n-m+1}+3}{2}$;
(ii) the diameter of $SWSG_{n,m}$ is $D_{n,m} = 2^{m-1} + 2$;
(iii) the clustering of $SWSG_{n,m}$ is $C_{n,m} = \frac{20\cdot 3^{n-2} - 2\cdot 3^{n-m} + 7}{5(3^n+5)}$.

Proof. The order of $SWSG_{n,m}$ is one plus the order of SG_n. The size of $SWSG_{n,m}$ is the size of SG_n, plus the number of added edges. Since the number of added edges is the order of SG_{n-m+1}, by proposition 1 we have $|V_{n,m}| = \frac{3^n+3}{2} + 1 = \frac{3^n+5}{2}$ and $|E_{n,m}| = 3^n + \frac{3^{n-m+1}+3}{2}$.

Let us denote by $D(SG_k)$ the diameter of SG_k and by $D_{n,m}$ the diameter of $SWSG_{n,m}$. To compute $D_{n,m}$ we need only observe that, in SG_m, every node is at distance at most $D(SG_m)/2$ from the set of vertices of the external triangle. An upper bound for $D_{n,m}$ is $D(SG_m) + 2 = 2^{m-1} + 2$. It can easily be seen that this is also a lower bound. Therefore, $D_{n,m} = 2^{m-1} + 2$.

The new node has clustering 0. The vertices of the external triangle are the only nodes of degree 3. These three nodes have clustering $\frac{1}{3}$. The remaining nodes have degree 4 or 5. For the nodes of degree 4, the clustering is either $\frac{1}{2}$ or $\frac{1}{3}$. Let x_n be the number of nodes of degree 4 with clustering $\frac{1}{2}$, and y_n be the number of nodes of degree 4 with clustering $\frac{1}{3}$. By construction of $SWSG_{n,m}$ and by proposition 1 we have

$$x_n = 3^{n-1}$$

and

$$y_n = \frac{3^{n-1} - 3}{2} - \frac{3^{n-m+1} - 3}{2} = \frac{3^{n-1} - 3^{n-m+1}}{2}.$$

The remaining $\frac{3^{n-m+1}-3}{2}$ nodes have degree 5 and clustering $\frac{1}{5}$.
This gives us the value for the clustering

$$C_n = \frac{\frac{1}{3}\cdot 3 + \frac{1}{2}\cdot 3^{n-1} + \frac{1}{3}\cdot \frac{3^{n-1}-3^{n-m+1}}{2} + \frac{1}{5}\cdot \frac{3^{n-m+1}-3}{2}}{\frac{3^n+5}{2}}$$
$$= \frac{20\cdot 3^{n-2} - 2\cdot 3^{n-m} + 7}{5(3^n+5)}.$$

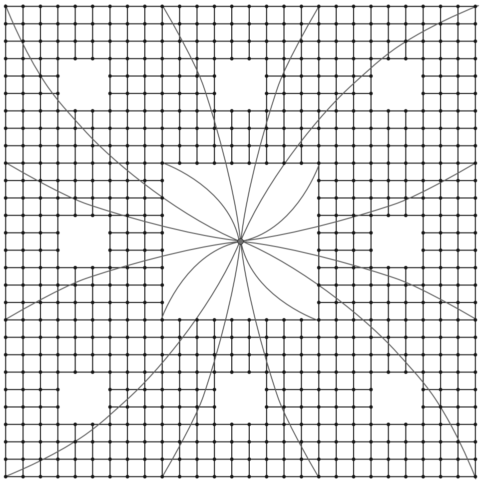

Figure 9. $SWSC_{4,3}$.

□

Corollary 8. *Let us denote by N and D the order and the diameter of* $SWSG_{n,m}$*, respectively. If* $m \leqslant \log_2 n$*, then* $D = O(\log N)$.

Proof. By proposition 7, $D = 2^{m-1} + 2$ and $N = \frac{3^n+5}{2}$. If $m \leqslant \log_2 n$, then $D \leqslant \frac{n}{2} + 2 = O(\log N)$. □

Proposition 9. *The box-counting dimension of* $SWSG_{n,m}$ *is the same as the box-counting dimension of* SG_n*, namely,* $d_B = \frac{\ln 3}{\ln 2} \approx 1.585$.

Proof. We can cover $SWSG_{n,m}$ with boxes of linear size ℓ_B in the same way as we cover SG_n (see proposition 2 and figure 3). We need only to add the new node to one of the boxes. We obtain exactly the same number of boxes for every size ℓ_B. So, the box-counting dimension of $SWSG_{n,m}$ and SG_n is the same. □

3.2. Small-world Sierpinski carpet

The small-world Sierpinski carpet, $SWSC_{n,m}$, is the graph defined as follows.

For every $n \geqslant 3$ and $m = 2, \ldots, n-1$, SC_n can be seen as 8^{n-m} copies of SG_m, with some path identifications. $SWSC_{n,m}$ is the graph obtained by joining a new node to every vertex of the external square of every copy of SC_m (see figure 9). The number of new edges is exactly the number of vertices of SC_{n-m+1}.

The diameter of this new graph depends on the value of m. We will show that, for some of the values of m, $D(SWSC_{n,m}) \approx \ln |V(SWSC_{n,m})|$. Thus, this construction leads to a graph which is a small-world fractal.

This definition is analogous to the definition of $SWSG_{n,m}$ (see section 3.1).

Next we give the properties of $SWSC$.

Proposition 10. *The small-world Sierpinski carpet,* $SWSC_{n,m}$*, satisfies the following properties:*

(i) the order and the size of $SWSC_{n,m}$ *are* $|V_{n,m}| = \frac{11}{70} \cdot 8^n + \frac{8}{5} \cdot 3^{n-1} + \frac{15}{7}$ *and* $|E_{n,m}| = \frac{3}{10} \cdot 8^n + \frac{8}{5} \cdot 3^{n-1} + \frac{11}{70} \cdot 8^{n-m+1} + \frac{8}{5} \cdot 3^{n-m} + \frac{8}{7}$;

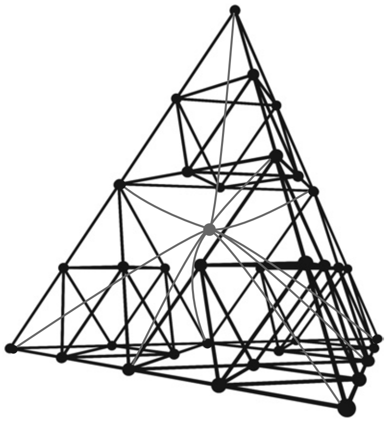

Figure 10. $SWST_{3,2}$.

(ii) the diameter of $SWSC_{n,m}$ *is* $D_{n,m} = 5 \cdot 3^{m-2} + 1$*;*
(iii) all the nodes in $SWSC_{n,m}$ *have clustering* 0*.*

Proof. The order of $SWSC_{n,m}$ is the order of SC_n, plus one. The size of $SWSC_{n,m}$ is the size of SC_n, plus the number of added edges. Since the number of added edges is the order of SC_{n-m+1}, by proposition 3 we have $|V_{n,m}| = \frac{11}{70} \cdot 8^n + \frac{8}{5} \cdot 3^{n-1} + \frac{8}{7} + 1 = \frac{11}{70} \cdot 8^n + \frac{8}{5} \cdot 3^{n-1} + \frac{15}{7}$ and $|E_{n,m}| = \frac{3}{10} \cdot 8^n + \frac{8}{5} \cdot 3^{n-1} + \frac{11}{70} \cdot 8^{n-m+1} + \frac{8}{5} \cdot 3^{n-m} + \frac{8}{7}$.

Let us denote by $\ell_k = 3^{k-1}$ the length of the side of the external square of SC_k (see the proof of proposition 3) and by $D_{n,m}$ the diameter of $SWSC_{n,m}$. To compute $D_{n,m}$ we need only observe that, in SC_m, every node is at distance at most $2\ell_{m-1} + \frac{\ell_{m-1}-1}{2} = \frac{5\ell_{m-1}-1}{2}$ from the set of vertices of the external square. An upper bound of $D_{n,m}$ is $5\ell_{m-1} - 1 + 2 = 5\ell_{m-1} + 1 = 5 \cdot 3^{m-2} + 1$. It can easily be seen that this is also a lower bound. Therefore, $D_{n,m} = 5 \cdot 3^{m-2} + 1$.

Since $SWSC_{n,m}$ has no triangles, the clustering is 0. □

Corollary 11. *Let us denote by N and D the order and the diameter of* $SWSC_{n,m}$*, respectively. If* $m \leqslant \log_3 n$*, then* $D = O(\log N)$*.*

Proof. By proposition 10, $D = 5 \cdot 3^{m-2} + 1$. If $m \leqslant \log_3 n$, then $D \leqslant \frac{5}{9}n + 1 = O(\log N)$. □

Proposition 12. *The box-counting dimension of* $SWSC_{n,m}$ *is the same as the box-counting dimension of SC, namely,* $d_B = \frac{\ln 8}{\ln 3} \approx 1.892$*.*

Proof. The proof is analogous to the proof of proposition 9, and uses the box counting shown in figure 5. □

3.3. Small-world Sierpinski tetra

The small-world Sierpinski tetra, $SWST_{n,m}$, is the graph defined as follows.

For every $n \geqslant 3$ and $m = 2, \ldots, n-1$, ST_n can be seen as 4^{n-m} copies of ST_m, with some node identifications. $SWST_{n,m}$ is the graph obtained by joining a new node to every vertex of the external tetrahedron of every copy of ST_m (see figure 10). The number of new edges is exactly the number of vertices of ST_{n-m+1}.

The diameter of this new graph depends on the value of m. We will show that, for some of the values of m, $D(SWST_{n,m}) \approx \ln|V(SWST_{n,m})|$. Thus, this construction gives a graph which is a small-world fractal.

This definition is analogous to the definition of $SWSG_{n,m}$ (see section 3.1) and $SWSC_{n,m}$ (see section 3.2).

Next we give the properties of $SWST$.

Proposition 13. *The small-world Sierpinski tetra, $SWST_{n,m}$, satisfies the following properties:*

(i) the order and the size of $SWST_{n,m}$ are $|V_{n,m}| = 2 \cdot 4^{n-1} + 3$ and $|E_{n,m}| = 6 \cdot 4^{n-1} + 2 \cdot 4^{n-m} + 2$;
(ii) the diameter of $SWST_{n,m}$ is $D_{n,m} = 2^{m-1} + 2$;
(iii) the clustering of $SWST_{n,m}$ is $C_{n,m} = \frac{35\cdot 4^{n-1} - 8\cdot 4^{n-m} - 50}{35(2\cdot 4^{n-1}+3)}$.

Proof. The order of $SWST_{n,m}$ is one plus the order of ST_n. The size of $SWST_{n,m}$ is the size of ST_n, plus the number of added edges. Since the number of added edges is the order of ST_{n-m+1}, by proposition 5 we have $|V_{n,m}| = 2(4^{n-1} + 1) + 1 = 2 \cdot 4^{n-1} + 3$ and $|E_{n,m}| = 6 \cdot 4^{n-1} + 2 \cdot 4^{n-m} + 2$.

Let us denote by $D(ST_k)$ the diameter of ST_k and by $D_{n,m}$ the diameter of $SWST_{n,m}$. To compute $D_{n,m}$ we need only observe that, in ST_m, every node is at distance at most $D(ST_m)/2$ from the set of vertices of the external tetrahedron. An upper bound of $D_{n,m}$ is $D(ST_m) + 2 = 2^{m-1} + 2$. It can easily be seen that this is also a lower bound. Therefore, $D_{n,m} = 2^{m-1} + 2$.

The new node has clustering 0. The vertices of the external tetrahedron are the only nodes of degree 4. Those four nodes have clustering $\frac{1}{2}$. The remaining nodes have degree 6 or 7. For the nodes of degree 6, the clustering is either $\frac{8}{15}$ or $\frac{6}{15}$. Let x_n be the number of nodes of degree 6 with clustering $\frac{8}{15}$, and y_n be the number of nodes of degree 6 with clustering $\frac{6}{15}$. By construction of $SWST_{n,m}$ and by proposition 5 we have

$$x_n = 6 \cdot 4^{n-2}$$

and

$$y_n = 2(4^{n-2} - 1) - 2(4^{n-m} - 1) = 2(4^{n-2} - 4^{n-m}).$$

The remaining $2(4^{n-m} - 1)$ nodes have degree 7 and clustering $\frac{6}{21}$.

This gives us the value for the clustering

$$C_n = \frac{\frac{1}{2} \cdot 4 + \frac{8}{15} \cdot 6 \cdot 4^{n-2} + \frac{6}{15} \cdot 2(4^{n-2} - 4^{n-m}) + \frac{6}{21} \cdot 2(4^{n-m} - 1)}{2 \cdot 4^{n-1} + 3}$$
$$= \frac{35 \cdot 4^{n-1} - 8 \cdot 4^{n-m} - 50}{35(2 \cdot 4^{n-1} + 3)}.$$
□

Corollary 14. *Let us denote by N and D the order and the diameter of $SWST_{n,m}$, respectively. If $m \leqslant \log_2 n$, then $D = O(\log N)$.*

Proof. By proposition 13, $D = 2^{m-1} + 2$ and $N = 2 \cdot 4^{n-1} + 3$. If $m \leqslant \log_2 n$, then $D \leqslant \frac{n}{2} + 2 = O(\log N)$. □

Proposition 15. *The box-counting dimension of $SWST_{n,m}$ is the same as the box-counting dimension of ST_n, namely $d_B = 2$.*

Proof. The proof is analogous to the proof of proposition 9, and uses the box counting shown in figure 7. □

4. Sierpinski versus small-world Sierpinski

In the previous section, we proposed a construction of small-world graphs for three families of graphs—the Sierpinski gasket, the Sierpinski carpet and the Sierpinski tetra. We have shown that all of these three families of graphs are fractal and have logarithmic diameter. To summarize, we have the following three cases:

- The small-world Sierpinski gasket, $SWSG_{n,m}$, is obtained from the Sierpinski gasket, SG_n, by adding $\frac{3^{n-m+1}+3}{2}$ edges. Its box-counting dimension is $\frac{\ln 3}{\ln 2} \approx 1.585$ and its diameter is $O(\log N)$, for $2 \leqslant m \leqslant \log_2 n$.
- The small-world Sierpinski carpet, $SWSC_{n,m}$, is obtained from the Sierpinski carpet, SC_n, by adding $\frac{11}{70} \cdot 8^{n-m+1} + \frac{8}{5} \cdot 3^{n-m} + \frac{8}{7}$ edges. Its box-counting dimension is $\frac{\ln 8}{\ln 3} \approx 1.892$ and its diameter is $O(\log N)$, for $2 \leqslant m \leqslant \log_3 n$.
- The small-world Sierpinski tetra, $SWST_{n,m}$, is obtained from the Sierpinski tetra, ST_n, by adding $2 \cdot 4^{n-m} + 2$ edges. Its box-counting dimension is 2 and its diameter is $O(\log N)$, for $2 \leqslant m \leqslant \log_2 n$.

As a measure of how the underlying structure is preserved in the construction of a small-world Sierpinski graph, we can evaluate the number of added edges and the clustering variation. In this sense, our construction will be better for large values of m, those values leading to diameter $O(\log N)$.

Next we show that both the number of added edges and the clustering variation are reasonably small. More precisely, for each of these three families, the optimal values of m have the property that the number of added edges and the clustering variation, with respect to the corresponding Sierpinski graph, are $O\left(\frac{N}{\log N}\right)$ and $O\left(\frac{1}{\log N}\right)$, respectively, where N denotes the order of the graph. For the sake of simplicity, we restrict our calculations to specific values of m.

Corollary 16. *If $m = \log_3 n$, then the size variation and the clustering variation between SG_n and $SWSG_{n,m}$ are $O\left(\frac{N}{\log N}\right)$ and $O\left(\frac{1}{\log N}\right)$, respectively, where N denotes the order of SG_n.*

Proof. By proposition 1, the order of SG_n is $N = \frac{3^n+3}{2}$. Moreover, its size is $E = 3^n$ and its clustering is $C = \frac{4 \cdot 3^{n-2}+5}{3^n+3}$. By proposition 7, the size of $SWSG_{n,m}$ is $E' = 3^n + \frac{3^{n-m+1}+3}{2}$ and its clustering is $C' = \frac{20 \cdot 3^{n-2} - 2 \cdot 3^{n-m} + 7}{5(3^n+5)}$.

Easy calculations give

$$E' - E = \frac{3^{n-m+1} + 3}{2}$$

and

$$\begin{aligned} C - C' &= \frac{4 \cdot 3^{n-2} + 5}{3^n + 3} - \frac{20 \cdot 3^{n-2} - 2 \cdot 3^{n-m} + 7}{5(3^n + 5)} \\ &= \frac{8 \cdot 3^{n-2} + 25}{(3^n + 3)(3^n + 5)} + \frac{2 \cdot 3^{n-m}}{5(3^n + 5)} - \frac{7}{5(3^n + 5)}. \end{aligned}$$

If $m = \log_3 n$, then $3^{n-m} = \frac{3^n}{n}$, which implies that the number of added edges is $E' - E = O\left(\frac{N}{\log N}\right)$, and the decreasing of the clustering is $C - C' = O\left(\frac{1}{\log N}\right)$. □

Corollary 17. *If $m = \log_3 n$, then the size variation between SC_n and $SWSC_{n,m}$ is $O\left(\frac{N}{\log N}\right)$, where N denotes the order of SC_n.*

Proof. By proposition 3, the order of SC_n is $N = \frac{11}{70} \cdot 8^n + \frac{8}{5} \cdot 3^{n-1} + \frac{8}{7}$ and its size is $E = \frac{3}{10} \cdot 8^n + \frac{8}{5} \cdot 3^{n-1}$. By proposition 10, the size of $SWSG_{n,m}$ is $E' = \frac{3}{10} \cdot 8^n + \frac{8}{5} \cdot 3^{n-1} + \frac{11}{70} \cdot 8^{n-m+1} + \frac{8}{5} \cdot 3^{n-m} + \frac{8}{7}$.

Easy calculations give

$$E' - E = \frac{11}{70} \cdot 8^{n-m+1} + \frac{8}{5} \cdot 3^{n-m} + \frac{8}{7}.$$

If $m = \log_3 n$, then $3^{n-m} = \frac{3^n}{n}$ and $8^{n-m} < \frac{8^n}{n}$, which implies that the number of added edges is $E' - E = O\left(\frac{N}{\log N}\right)$. □

Remark. In this case, the discussion about the clustering coefficient is not needed, because its value is 0 for both graphs SC_n and $SWSC_{n,m}$.

Corollary 18. *If $m = \log_4 n$, then the size variation and the clustering variation between ST_n and $SWST_{n,m}$ are $O\left(\frac{N}{\log N}\right)$ and $O\left(\frac{1}{\log N}\right)$, respectively, where N denotes the order of ST_n.*

Proof. By proposition 5, the order of ST_n is $N = 2(4^{n-1} + 1)$. Moreover, its size is $E = 6 \cdot 4^{n-1}$ and its clustering is $C = \frac{5\cdot 4^{n-1}+16}{10(4^{n-1}+1)}$. By proposition 13, the size of $SWSG_{n,m}$ is $E' = 6 \cdot 4^{n-1} + 2 \cdot 4^{n-m} + 2$ and its clustering is $C' = \frac{35\cdot 4^{n-1} - 8\cdot 4^{n-m} - 50}{35(2\cdot 4^{n-1}+3)}$.

Easy calculations give

$$E' - E = 2 \cdot 4^{n-m} + 2$$

and

$$C - C' = \frac{5 \cdot 4^{n-1} + 16}{10(4^{n-1} + 1)} - \frac{35 \cdot 4^{n-1} - 8 \cdot 4^{n-m} - 50}{35(2 \cdot 4^{n-1} + 3)}$$
$$= \frac{37 \cdot 4^{n-1} + 48}{10(4^{n-1} + 1)(2 \cdot 4^{n-1} + 3)} + \frac{8 \cdot 4^{n-m}}{35(2 \cdot 4^{n-1} + 3)} + \frac{10}{7(2 \cdot 4^{n-1} + 3)}.$$

If $m = \log_4 n$, then $4^{n-m} = \frac{4^n}{n}$, which implies that the number of added edges is $E' - E = O\left(\frac{N}{\log N}\right)$, and the decreasing of the clustering is $C - C' = O\left(\frac{1}{\log N}\right)$. □

5. Conclusions

We have presented the properties of Sierpinski graphs, including their clustering and fractality. We have also proposed a deterministic construction of small-world Sierpinski graphs, and studied their properties. We have applied combinatorial techniques to show that the structure of the Sierpinski graphs is preserved, including fractality, while the small-world phenomenon arises.

In our small-world constructions, the added node clearly acts as a hub. For communication purposes, it would have to be improved. However, our goal was to show that the box-counting method, which is used to compute the classical box-counting dimension and has been adapted for graphs in [6], works even in a small-world network. In this sense, we can say that we have found a family of fractal, small-world deterministic networks, albeit that the definition of fractality in networks may not yet be standard.

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